Cayley graphs. Normal subgroups

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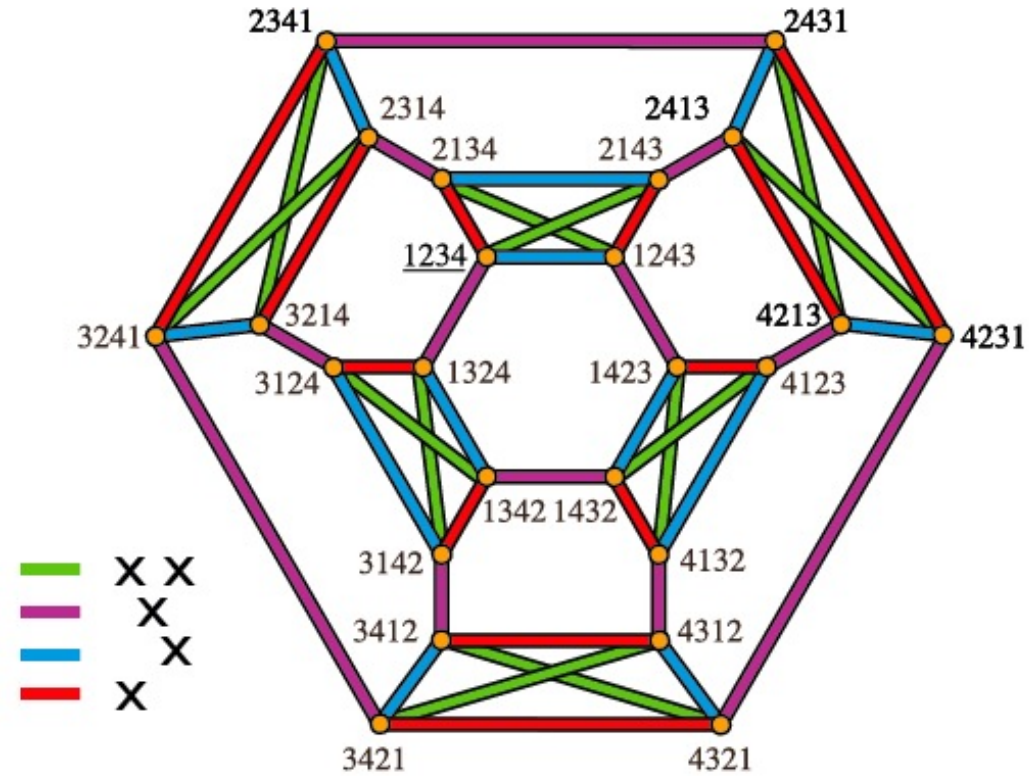
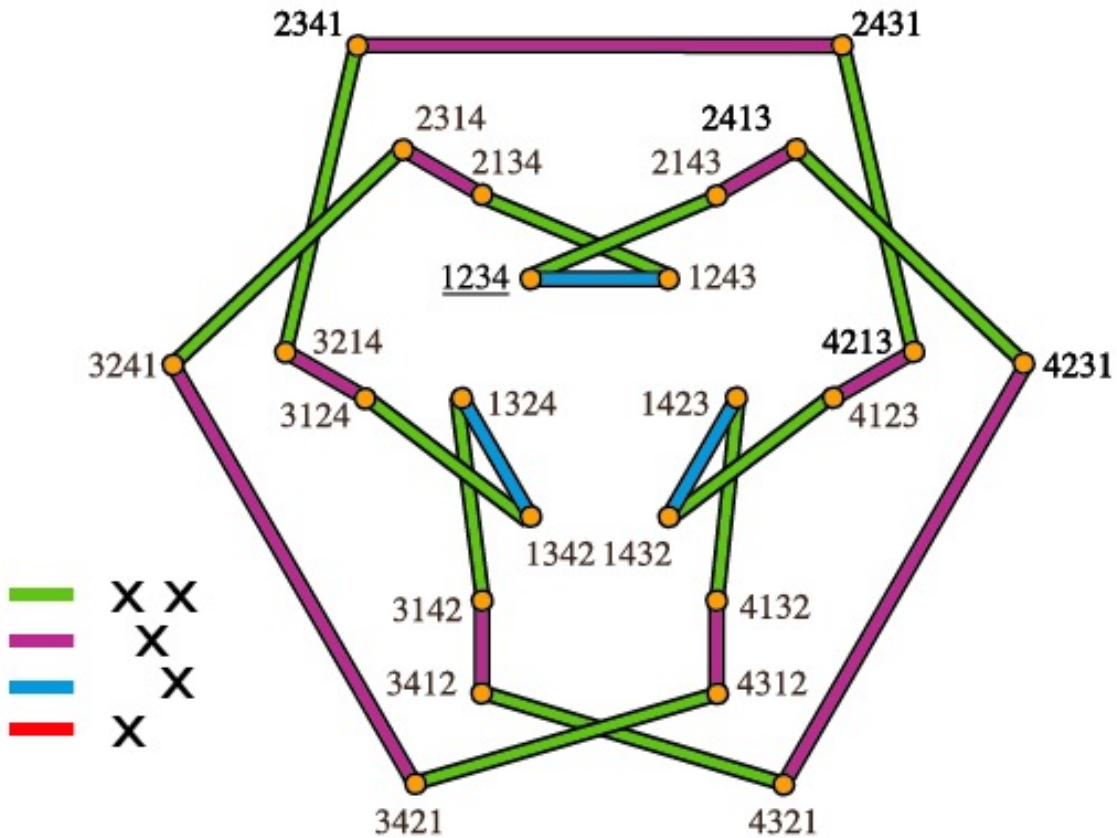
Graph out of bell ringing

- Think of permutations as **vertices of a graph**.
- Two vertices are connected by an edge if there is a permitted transition (according to bell ringers) that transforms one change into the other. Here what it looks like for 4 bells:



Hamiltonian cycle

- An extent is a path in this graph, visiting each of the vertices exactly once, and returning to the beginning vertex. Such tours are called **Hamiltonian cycles**.
- For Plain Bob, this path looks like that:



Cayley Graph

Let G be a group, and let S be a generating set of elements.

Definition

Let $Cay(G, S)$ be the colored directed graph having G as the set of vertices, and for any $s \in S$ there is an edge going from g to gs, and any such edge is colored into a unique color c_s corresponding to $s \in S$.

- Draw Cayley graphs for $\mathbb{Z}$ with $S_1 = \{1\}$, and with $S_2 = \{2, 3\}$.
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- D_4 with generators r_{90} (rotation by 90°) and s_h (vertical reflection)?

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- Prove that any Cayley graph is connected (if we ignore the orientation of edges).
- Between any two vertices g, h there is at most one edge.
- All vertices have the same degrees.
- What do (un-oriented) cycles in Cayley graphs mean?
- Any group acts on its Cayley graph, sending a vertex corresponding to h to the vertex corresponding to gh.

Group from its Cayley graph

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- We then define $[a] * [b]$ to be the equivalence class of the composite path, first going along a_0, then continuing as $g_0 s_{i_1}$, $g_0 s_{i_1} s_{i_2}$ etc. all the way up to $g_0 h_0$.

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- It's also injective, so we get an isomorphism. **Done.**

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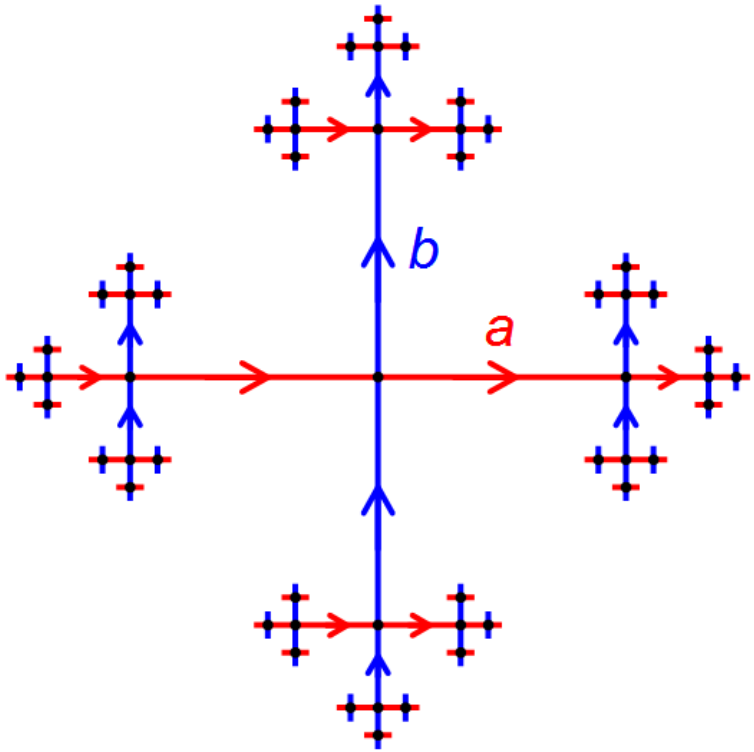

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- The subgroup A_n of even permutations is normal in S_n.

Cosets for normal subgroups

The point is: if H is normal, the set of cosets G/H has a natural group structure. This group is called the **quotient group**.

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- $\mathbb{R}/\mathbb{Z}$ is a circle S^1.

Semidirect product

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- If G acts trivially on K, then $G \ltimes K \simeq G \times K$.
- The group D_{2n} of symmetries of the n-gon is $\mathbb{Z}/2 \ltimes \mathbb{Z}/n$, where the action of $\mathbb{Z}/2$ on $\mathbb{Z}/n$ is by $a \mapsto -a$.

Semidirect product

Theorem

Let G be a group, and H, K are two subgroups. Suppose that

- $H \cap K = \{e\}$;
- $G = KH$ *as a set;*
- K *is a normal subgroup of* G.

Then $G \simeq H \ltimes K$, where the action of H on K is given by conjugation ${}^h k = hkh^{-1}$.